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Polynomial Chaos-Based Analysis of Probabilistic Uncertainty in Hypersonic Flight Dynamics

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In this paper, we present a novel computational framework for analyzing the evolution of the uncertainty in state trajectories of a hypersonic air vehicle due to the uncertainty in initial conditions and other system parameters. The framework is built on the so-called generalized polynomial chaos expansions. In this framework, stochastic dynamical systems are transformed into equivalent deterministic dynamical systems in higher dimensional space. Here, the evolution of uncertainty due to initial condition, ballistic coefficient, lift over drag ratio, and atmospheric density is analyzed. The problem studied here is related to the Mars entry, descent, and landing problems. We demonstrate that the polynomial chaos framework is able to predict evolution of uncertainty, in hypersonic flight, with the same order of accuracy as the Monte–Carlo methods but with more computational efficiency.

Nomenclature

B_c	=	ballistic coefficient of the vehicle = 72.8 kg/m^2
$\mathcal{F}$	=	σ algebra defined over subsets of Ω
$f(\Delta)$	=	joint probability density function of system parameters
h	=	altitude, km
h_1	=	scale height in density computation = 9.8×10^3 m
h_2	=	scale height in density computation = 20×10^3 m
P	=	probability measure
R_0	=	radius of Mars = 3397 km
t	=	time, s
v	=	velocity, km/s
γ	=	flight-path angle, rad
$\Delta \equiv \Delta(\omega)$	=	random variables representing system parameters
ν	=	lift over drag ratio = 0.3
ρ_0	=	density on the surface of Mars = 0.0176 kg/m^3
Ω	=	sample space
$\omega \in \Omega$	=	event in the sample space

I. Introduction

NASA has clearly identified the need for fundamental research on entry, descent, and landing of large robotic and manned spacecraft on the surface of Mars with high accuracy. Previous missions have delivered less than 0.6 metric tons (MT) and had large uncertainty in the landing location (100 s of km). The Mars exploration program has attempted to develop systems that would deliver 0.7 MT for the Mars Science Laboratory and 1.2 MT for the Mars sample return mission. NASA has expressed a clear need to develop entry/descent/landing systems that can get 30 to 100 MT down to the surface per landing. These missions included plans for a precision guided entry and a tether-based payload deployment system, which was expected to provide a landing accuracy of 20–40 km from the target. Another major concern with high-mass entry was the mismatch between the entry conditions and the deceleration capabilities provided by supersonic parachute technologies. In such applications, there was uncertainty present in the initial condition and other system parameters. Hence, for a successful mission, it was critical to study the impact of such uncertainties on state trajectories and determine the uncertainty in the landing site and the entry condition for supersonic parachute deployment.

Figure 1 summarizes the research problem addressed in this paper. Figure 1a illustrates the effect of uncertainty in the initial condition on the landing site. Figure 1b illustrates the uncertainty in the landing site for various guidance solutions with different parameters. Thus, future missions to Mars require a rigorous analysis of the uncertainties in the system parameters on the trajectory and the landing site of the vehicle, and they serve as the motivation for this work.

The problem of studying the evolution of uncertainty in dynamical systems has consistently been of interest in the scientific community. In many cases, the uncertainties in the initial condition and other system parameters for a dynamical system are not known precisely. However, characteristics of the uncertainties, such as mean, standard deviation, probability density function, etc., may be known.

State-of-the-art algorithms for uncertainty propagation often assume that the system can be modeled as a linear Gaussian process. Given a linear system of $\dot{x} = Ax$, $x \in \mathbb{R}^n$, with the initial Gaussian condition uncertainty defined by the covariance matrix P and the mean $\bar{x}$, the evolution of state uncertainty is given by the well-known equations $\dot{P} = A^T P + PA$ and $\dot{\bar{x}} = A\bar{x}$. For non-Gaussian distributions, the density function is often approximated using Gaussian mixtures [1], and the uncertainty in the state trajectories is computed by superposing evolutions of every approximating Gaussian distribution. For nonlinear dynamical systems, there are several methods to determine uncertainty in state trajectories due to stochastic forcing or parameters. Some of the popular nonlinear methods include Monte–Carlo (MC) [2], Markov chain MC (MCMC) [2], local linearization [3], unscented Kalman filtering (UKC) [4], Fokker–Planck–Kolmogorov framework [5], and cell-to-cell mapping [6].

It is well known that MC methods are not computationally scalable and suffer from statistical inconsistencies. They are still the most popular method because of easy implementation. In MCMC, a Markov chain is constructed (via intelligent sampling) that has the desired property of the random process being studied. The sampling in this case is correlated and not independent, as it is in classical MC. Construction of the associated Markov chain is relatively easy; however, MCMC methods have convergence issues. More sophisticated techniques, which include coupling from the past, provide better results but with additional computation and unbounded running time. Sequential MC (SMC) [7] is the sequential

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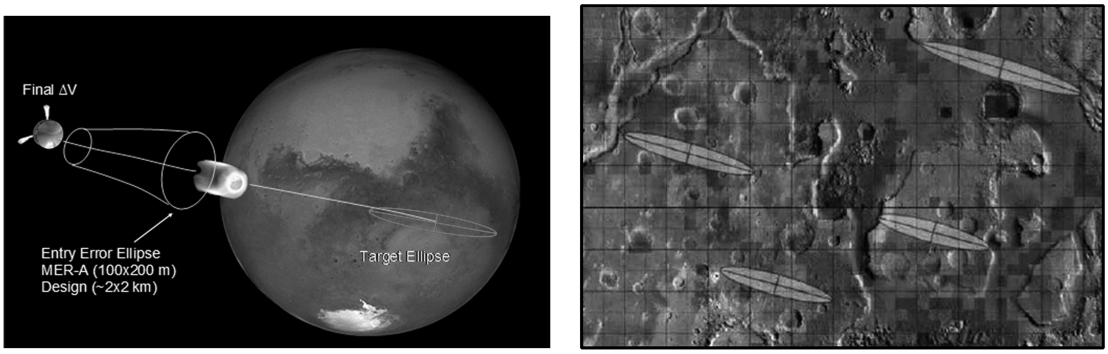


a) Uncertainty in initial condition and its impact on state trajectories

b) Uncertainty in the landing site

Fig. 1 Uncertainty in Mars exploration rover A (MER-A) evolution in hypersonic flight dynamics (Image: courtesy of NASA).

analogue of MCMC, which is similar to the importance of the sampling methods [8]. If well designed, SMC can be much faster than MCMC. The techniques MC, MCMC, and SMC do not assume Gaussian behavior of the random process and, with sufficiently large samples, they approach the Bayesian optimal estimate.

If the nonlinear random process exhibits Gaussian behavior, then the nonlinear system is linearized locally about the current mean and covariance, and the uncertainty is propagated using the locally approximated linear dynamics. This method is used in extended Kalman filters (EKF) [3]. It is well known that this approach performs poorly when nonlinearities are high [9]. The error in mean and covariance can be reduced if the uncertainty is propagated, using the nonlinear dynamics, for a minimal set of sample points (called sigma points). The density function of the state is parameterized by the sigma points, which completely captures the true mean and covariance. This approximated density function, when propagated through the true nonlinear system, captures the posterior mean and covariance accurately to the third order (Taylor series expansion) for any nonlinearity with Gaussian behavior. This technique is used in UKFs [4]. The computational complexity of the UKF is the same order as that of the EKF. The methods described so far compute the evolution of the probability density function in the state trajectory due to nonlinear dynamics, uncertain system parameters, and stochastic forcing. Methods based on higher order approximation of nonlinear dynamics have also been proposed. These require tensor calculus resulting in higher computational overhead [10].

In this research effort, we are interested in developing methods for uncertainty propagation based on polynomial chaos (PC) expansions, which can be thought of as an extension of Volterra's theory of nonlinear functionals for stochastic systems [11]. This framework provides a computationally efficient approach for computing evolution of uncertainty for non-Gaussian nonlinear systems. In this framework, an approximation of the solution is assumed with unknown time-varying coefficients, which are determined by solving a set of deterministic differential equations resulting from either the Galerkin projection [12] or the stochastic collocation [13]. This approach requires preprocessing to derive the deterministic differential equations, which have to be performed once for a given system. The evolution of uncertainty is then determined by solving the obtained set of ordinary differential equations.

PC was first introduced by Wiener [14] for which Hermite polynomials were used to model stochastic processes with Gaussian random variables. According to Cameron and Martin [15], such an expansion converges in the $\mathcal{L}_2$ sense for any arbitrary stochastic process with finite second moment. This applies to most physical systems. Xiu and Karniadakis [16] generalizes the result of Cameron–Martin to various continuous and discrete distributions, using orthogonal polynomials from the so-called Askey scheme [17] and demonstrating $\mathcal{L}_2$ convergence in the corresponding Hilbert functional space. This is popularly known as the generalized PC (GPC) framework. The GPC framework has been applied to various applications, including stochastic fluid dynamics [18,19], stochastic finite elements [12], and solid mechanics [20,21]. Application of GPC to problems related to control and estimation of dynamical systems has been surprisingly limited.

The paper is organized as follows. Preliminaries on the theory of PC and demonstrated transformation of stochastic dynamics, with parametric uncertainty, into deterministic dynamics in higher dimensional state space is presented. This is followed by the characterization of the stochastic Vinh's equation [22] for longitudinal motion, for which we assume uncertainty in ballistic coefficient, lift over drag ratio, density parameters, and initial conditions. The stochastic hypersonic dynamics is then transformed into deterministic dynamics in higher dimensional state space using PC expansions. This is followed by numerical results that characterize uncertainty propagation in state trajectories induced by uncertainty in system parameters. The results obtained from the PC framework are compared with the MC simulations, which are shown to agree well.

II. Polynomial Chaos

A. Generalized Polynomial Chaos

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ algebra of the subsets of Ω, and P is the probability measure. Let

$$\Delta(\omega) = [\Delta_1(\omega), \cdots, \Delta_d(\omega)]\colon (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}^d)$$

be an $\mathbb{R}^d$-valued continuous random variable, where $d \in \mathbb{N}$ and $\mathcal{B}^d$ are the σ algebra of Borel subsets of $\mathbb{R}^d$. A general second-order process $X(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ can be expressed by PC as

$$X(\omega) = \sum_{i=0}^{\infty} x_i \phi_i[\Delta(\omega)] \tag{1}$$

where ω is the random event and $\phi_i[\Delta(\omega)]$ denotes the GPC basis of degree i in terms of the random variables $\Delta(\omega)$. The functions $\{\phi_i\}$ are a family of orthogonal basis in $\mathcal{L}_2(\Omega, \mathcal{F}, P)$, satisfying the relation:

$$\mathbf{E}[\phi_i \phi_j] = \mathbf{E}[\phi_i^2]\delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta, and $\mathbf{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathrm{d}P(\omega) = f[\Delta(\omega)]\,\mathrm{d}\omega$ and probability density function $f[\Delta(\omega)]$. Henceforth, Δ will be used to represent $\Delta(\omega)$.

For random variables Δ with certain distributions, the family of orthogonal basis functions $\{\phi_i\}$ can be chosen in such a way that its weight function has the same form as the probability density function $f(\Delta)$. These orthogonal polynomials are members of the Askey scheme of polynomials [17], which form a complete basis in the Hilbert space determined by their corresponding support. Table 1 summarizes the correspondence between the choice of polynomials for a given distribution of Δ [16].

B. Approximate Solution of Stochastic Differential Equations

A stochastic dynamical system of the form $\dot{x} = f(x, \Delta)$, where $x \in \mathbb{R}^n$, $\Delta \in \mathbb{R}^d$ can be solved using the PC framework, as shown in the following. Assume the solution of the stochastic differential equation to be $x(t, \Delta)$. For second-order processes, the solution for every component of $x \in \mathbb{R}^n$ can be approximated as

$$\hat{x}_i(t, \Delta) = \sum_{j=0}^{P} x_{ij}(t)\phi_j(\Delta); \qquad i = 1, \cdots, n \tag{3}$$

Substituting the approximate solution into the dynamical system results in errors:

$$e_i = \dot{\hat{x}}_i - f_i(\hat{x}, \Delta); \qquad i = 1, \cdots, n$$

The approximation in Eq. (3) is optimal in the $\mathcal{L}_2$ sense when the projections of the error on the orthogonal basis functions are zero, that is,

$$\begin{gathered}\langle e_i(t, \Delta), \phi_j(\Delta)\rangle = \int_{\mathcal{D}_\Delta} e_i(t, \Delta)\phi_j(\Delta) f(\Delta)\, \mathrm{d}\Delta = 0 \\ j = 0, \cdots, P; \qquad i = 1, \cdots, n\end{gathered} \tag{4}$$

where $\mathcal{D}_\Delta$ is the domain of Δ. Equation (4) results in $n(P+1)$ deterministic ordinary differential equations in terms of $\hat{x}_i$ and $\dot{\hat{x}}_i$, which can be solved numerically to obtain the approximated stochastic response. Therefore, the stochastic dynamics in $\mathbb{R}^n$ have been transformed into deterministic dynamics in $\mathbb{R}^{n(P+1)}$. The series is truncated after $P+1$ terms, which is determined by the dimension d of Δ and the order r of the orthogonal polynomials $\{\phi_j\}$, satisfying $P + 1 = [(\mathrm{d} + r)!]/(\mathrm{d}!r!)$. This expression gives the number of terms in a sequence of multivariate polynomials up to order r with d variables.

To demonstrate the application of this theory, consider a first-order linear system:

$$\dot{x} = ax \tag{5}$$

where a is a random variable with uniform distribution in $[-1, 0]$. From Table 1, the appropriate basis functions for this distribution are Legendre polynomials. However, Legendre polynomials are defined over $[-1, 1]$. Therefore, we represent the random variable a as $a(\Delta) = -0.5 + 0.5\Delta$, where $\Delta \in [-1, 1]$. The random variable can also be written in terms of PC expansions as

$$a(\Delta) = \sum_{i=0}^{p} a_i\phi_i(\Delta)$$

with $a_0 = -0.5$, $a_1 = 0.5$, and $a_i = 0$ for $i > 1$. Note that in this case, the PC expansion for $a(\Delta)$ is exact. Representing the solution of the stochastic differential equation as

$$\hat{x}(t, \Delta) = \sum_{i=0}^{p} x_i(t)\phi_i(\Delta)$$

To obtain the stochastic differential equation for $x_i(t)$, the approximate solution is substituted in the dynamics, and the error is projected on the basis functions. The residue is given by

Table 1 Correspondence between choice of polynomials and given distribution of $\Delta(\omega)$

Random variable Δ	$\phi_i(\Delta)$ of the Askey scheme
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

$$e(t, \Delta) = \dot{\hat{x}}(t, \Delta) - a(\Delta)\hat{x}(t, \Delta) = \sum_{i=0}^{p} \dot{x}_i(t)\phi_i(\Delta) - \sum_{i=0}^{p}\sum_{j=0}^{p} a_i x_j(t)\phi_i(\Delta)\phi_j(\Delta)$$

Projecting $e(t, \Delta)$ on $\{\phi_k\}$ and setting it to zero, we get

$$\langle e(t, \Delta)\phi_k(\Delta)\rangle = \int_{-1}^{1} e(t, \Delta)\phi_k(\Delta) f(\Delta)\, \mathrm{d}\Delta = 0$$

for $k = 0, 1, \ldots, p$. This gives us

$$\langle \dot{\hat{x}}(t, \Delta)\phi_k(\Delta)\rangle = \langle a(\Delta)\hat{x}(t, \Delta)\phi_k(\Delta)\rangle$$

or

$$\sum_{i=0}^{p} \dot{x}_i(t)\langle \phi_i(\Delta)\phi_k(\Delta)\rangle = \sum_{i=0}^{p}\sum_{j=0}^{p} x_i(t)a_j\langle \phi_i(\Delta)\phi_j(\Delta)\phi_k(\Delta)\rangle$$

Because $\{\phi_k\}$ are orthogonal, this implies

$$\dot{x}_k(t) = \left[\sum_{i=0}^{p}\sum_{j=0}^{p} x_i(t)a_j\langle \phi_i(\Delta)\phi_j(\Delta)\phi_k(\Delta)\rangle / \langle \phi_k^2(\Delta)\rangle\right] \tag{6}$$

for

$$k = 0, 1, \ldots, p$$

which can be written in a compact form as

$$\dot{X}_{pc} = A_{pc}X_{pc}$$

where the matrix $A_{pc} \in \mathbb{R}^{(p+1)\times(p+1)}$ is determined from Eq. (6) and $X_{pc} = [x_0 \quad x_1 \quad \cdots \quad x_p]^T$. Therefore, we observe that the stochastic dynamics in $x(t, \Delta) \in \mathbb{R}$ are transformed to deterministic dynamics in $X_{pc}(t) \in \mathbb{R}^{p+1}$, which can be solved using standard numerical algorithms for ordinary differential equations. The transformation also retains linearity in the dynamics. Generalized expressions for $x(t, \Delta) \in \mathbb{R}^n$, with linear dynamics, have been derived in [23].

For the first-order linear system in Eq. (5), analytical results, or mean and variance of (x_t, Δ), exist, and the comparison of results from the PC theory and analytical results are discussed in Sec. VIII.

III. Propagation of Uncertainty in Linear Systems

In this section, the PC framework is used to develop a mechanism for propagating uncertainty in state trajectories, due to uncertainty in initial condition, for linear systems. For the PC theory to be applicable, we assume that the probabilistic uncertainty has finite second moment. This is true for most engineering systems. Assume the linear system dynamics of the form

$$\dot{x} = Ax \tag{7}$$

The initial condition of x is assumed to be uncertain with given probability density function. As a result, the state trajectory $x(t)$ is stochastic. Generally for linear systems, uncertainty in state is propagated in the following manner when there is no parametric uncertainty or stochastic forcing. Let $\bar{x} = \mathbf{E}[x]$ and let $\mathbf{E}\{[x(t) - \bar{x}(t)][x(t) - \bar{x}](t)\} = P(t)$. Then we can write the evolution of the covariance matrix $P(t)$ by taking its derivative in the following manner:

$$\begin{aligned}\dot{P} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[(x - \bar{x})(x - \bar{x})^T] = \mathbf{E}[(\dot{x} - \dot{\bar{x}})(x - \bar{x})^T] \\ &\quad + \mathbf{E}[(x - \bar{x})(\dot{x} - \dot{\bar{x}})^T] \\ \dot{P} &= AP + PA^T\end{aligned} \tag{8}$$

Similar expressions can be obtained to compute evolution of higher order moments, but we will confine our discussion to the second moment. To perform this same operation in the GPC framework, note that the uncertain initial condition $x(0,\ \Delta)$ can be written as

$$x(0,\ \Delta) = x_0 + \boldsymbol{\sigma}\boldsymbol{\Delta} \tag{9}$$

where $\boldsymbol{\sigma}$ is a matrix related to the covariance of the initial condition and $\boldsymbol{\Delta}$ is a vector of independent random variables:

$$\boldsymbol{\Delta} = [\,\Delta_1,\ \Delta_2,\ \ldots\ ,\ \Delta_d\,] \tag{10}$$

Here, d corresponds to the number of independent random variables governing the initial condition uncertainty. If all initial conditions are governed by independent random variables, then

$$\boldsymbol{\sigma} = \operatorname{diag}\left(\frac{\sigma_{x_1}}{\langle\Delta_1^2\rangle},\ \frac{\sigma_{x_2}}{\langle\Delta_2^2\rangle},\ \ldots\ ,\ \frac{\sigma_{x_n}}{\langle\Delta_d^2\rangle}\right) \tag{11}$$

where $\sigma_{x_i}^2$ is the variance in the initial condition with respect to the initial value of x_i, and $\langle\Delta_j^2\rangle$ corresponds to the inner product of Δ_j with itself. For a general initial condition distribution with covariance P_0, $\boldsymbol{\sigma}$ can be determined from

$$\begin{aligned} P_0 &= \mathbf{E}\{[x(0) - \bar{x}(0)][x(0) - \bar{x}(0)]^T\} = \mathbf{E}[\boldsymbol{\sigma}\boldsymbol{\Delta}\boldsymbol{\Delta}^T\boldsymbol{\sigma}^T] \\ &= \boldsymbol{\sigma}\mathbf{E}[\boldsymbol{\Delta}\boldsymbol{\Delta}^T]\boldsymbol{\sigma}^T \end{aligned} \tag{12}$$

where

$$\mathbf{E}\,[\boldsymbol{\Delta}\boldsymbol{\Delta}^T] = \operatorname{diag}(\langle\Delta_1^2\rangle,\ \langle\Delta_2^2\rangle,\ \ldots\ ,\ \langle\Delta_d^2\rangle)$$

This matrix is diagonal because of the independence of each random variable Δ_j. This expression gives us insight into how the statistics of the initial conditions are related to GPC expansion. If P_0 is written in terms of its spectral decomposition, we obtain

$$P_0 = M\Lambda M^T = \boldsymbol{\sigma}\mathbf{E}[\boldsymbol{\Delta}\boldsymbol{\Delta}^T]\boldsymbol{\sigma}^T \tag{13}$$

where $M \in \mathbb{R}^{n\times d}$ is a matrix of eigenvectors corresponding to the nonzero eigenvalues of P_0, and Λ is a diagonal matrix of the nonzero eigenvalues of P_0. Clearly, $\operatorname{rank}(P_0) = d$. As a result, $\boldsymbol{\sigma}$ is related to the eigenvectors and eigenvalues of P_0 in the following manner:

$$\boldsymbol{\sigma} = M\Lambda^{1/2}\mathbf{E}[\boldsymbol{\Delta}\boldsymbol{\Delta}^T]^{-1/2} \tag{14}$$

To propagate the uncertainty in the system using PC, the states must be written in terms of the GPC expansion as

$$x(t) = \sum_{k=0}^{P} x_k(t)\phi_k(\boldsymbol{\Delta}) \tag{15}$$

Following the methodology of Sec. II.B, this expression is substituted into the equations of motion. Because all of the other terms (A, B, and u) are deterministic, these do not have to be expanded (their expansions consist of a single constant term). Substituting into the equations gives

$$\sum_{k=0}^{P} \dot{x}_{k,i}\phi_k = A_i\left(\sum_{j=0}^{P} x_j\phi_j\right) + B_i u \tag{16}$$

where $x_{k,i}$ represents the kth term in the GPC expansion for the ith state, A_i represents the ith row of A, and B_i represents the ith row of B. Performing the Galerkin projection and setting the residue to zero yields

$$\dot{\mathbf{X}} = (A \otimes I_{P+1})\mathbf{X} + (B \otimes I_{P+1})\mathbf{U}_0 \tag{17}$$

where I_{P+1} is the identity matrix of dimension $P + 1$ and

$$\mathbf{X} = [\,x_{0,1}\quad x_{0,2}\quad \cdots\quad x_{0,n}\quad x_{1,1}\quad x_{1,2}\quad \cdots\quad x_{P,n}\,]^T \tag{18}$$

$$\mathbf{U}_0 = u \otimes [\,1\quad 0\quad \cdots\quad 0\,]^T \tag{19}$$

The symbol $\otimes$ in the preceding equations denotes the Kronecker product. The structure of the resulting equations is a decoupled system of $P + 1$ linear equations with n variables. The system structure implies that each state vector x_k associated with the polynomial ϕ_k evolves independently of all other vectors x_j, where $j \neq k$. This means that, for linear systems, there is no mixing between the vectors of coefficients (states) associated with each of the different ϕ_k. In other words, the evolution of coefficients of ϕ_0 can never influence those associated with ϕ_1, and so forth. As a result, only the evolution of the nonzero coefficients in the initial condition is required to predict the statistics. For an initial condition of the form of Eq. (9), for example, we only need to propagate $d + 1$ sets of coefficients. In general, if the initial condition uncertainty is linear in $\boldsymbol{\Delta} \in \mathbb{R}^d$, the evolution of the statistics can be captured exactly with the evolution of $n(d + 1)$ states. With the traditional approach, $[n(n + 1)]/2]$ terms are needed to capture the second-order moments, and an additional n term is needed to describe the evolution of the mean. For processes that can be completely described by knowledge of the mean and variance with $n = d$, the methods are of the same order. If the process has a nonzero skewness, then additional terms will be needed to directly determine the evolution of the third-order moments. Using GPC, this is not the case, as the statistics are represented exactly by the expansion, and the evolution of the GPC coefficients can in turn be used to determine the desired moments.

As an example, we will consider the propagation of initial condition uncertainty for the following linear system:

$$\dot{x} = Ax$$

where

$$A = \begin{bmatrix} -0.4972 & 2.1793 & 0.5056 \\ 0.4714 & -0.5664 & 0.8351 \\ -0.1087 & -5.6978 & 0.0636 \end{bmatrix}$$

and $x = [\,x_1\quad x_2\quad x_3\,]^T$. The Eigenvalues of this system are -1, $2i$, and $-2i$. These have been chosen to ensure that the uncertainty magnitude does not decay to zero over time. To validate the use of the GPC expansion for the propagation of the initial condition uncertainty in this system, we will first consider uncertainty in all three initial conditions, for which the initial uncertainty is assumed to be Gaussian, with mean $\bar{x} = [\,1\quad 1\quad 1\,]^T$ and unit covariance. If the process is Gaussian, the mean and covariance of the state completely describes the process. We will therefore compare the mean and variance trajectories generated by the GPC expansion to those of the traditional linear approach. As expected, the predicted mean and variance responses are identical (to integration accuracy) for GPC and the traditional linear propagation. As a demonstration, the mean and variance of the state x_1 are shown in Fig. 2. As the figure demonstrates, the GPC expansion is able to accurately predict the resulting system behavior. In fact, as mentioned previously, this prediction is exact when the proper set of orthogonal polynomials is used.

For initial condition uncertainty that can be completely described by mean and covariance, the GPC expansion (although useful) is no better than the usual method of propagating the covariance directly. When the uncertainty in the initial conditions is, however, governed by a process that cannot be completely described by mean and variance, higher order tensors will need to be propagated to capture the resulting behavior. With the GPC approach, this is not the case. For example, if the uncertainty in the previous example was governed by a beta distribution with $\alpha = 2$ and $\beta = 5$, the probability distribution function will have a shape, as shown in Fig. 3.

This probability density function clearly has nonzero skewness and will require propagation of the third moment. Using the GPC method, however, it is possible to propagate the statistics exactly by only propagating the same nd states. For this example, we will consider uncertainty only in x_1 of the form $x_1(0,\ \Delta) = 1 + \Delta$, where Δ is governed by the density function, shown in Fig. 3. The mean of $x_1(0,\ \Delta)$ is not one because the probability density function is not

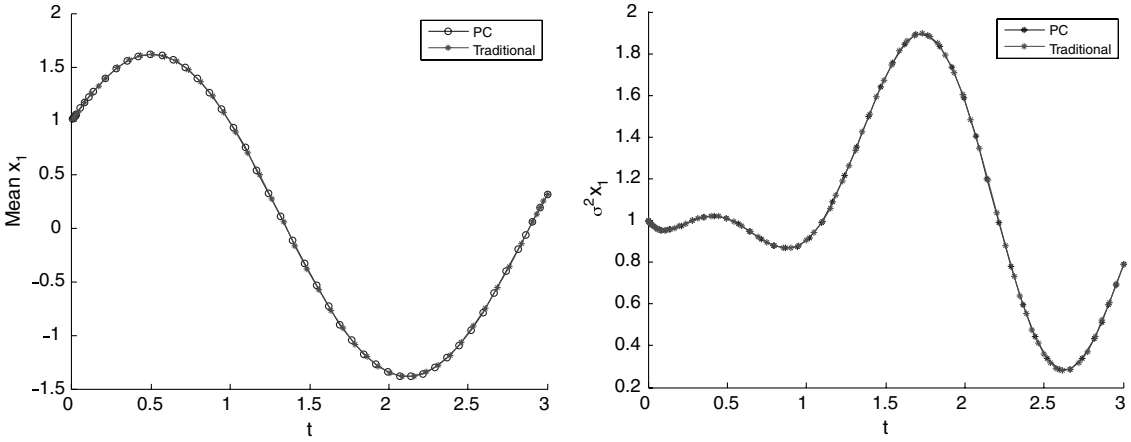


Fig. 2 Predicted mean and variance for a linear system using a traditional approach and GPC.

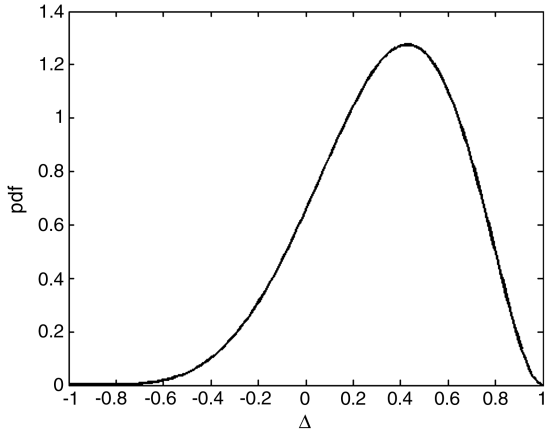


Fig. 3 Probability density function for beta distribution with $\alpha = 2$ and $\beta = 5$.

symmetric. The ability of the GPC to capture this behavior will be demonstrated using an expansion of two terms that are orthogonal with respect to the given beta distribution. Figure 4 shows the mean and variance of x_1 as predicted by GPC and MC methodologies. Again, the GPC expansion demonstrates very good accuracy that is to be expected, as it is exact for this application. This method can be used to accurately propagate uncertainty for linear systems for any type of probability distribution and also for cases in which initial conditions are functions of the variables of multiple distributions.

Next, a flight control problem is considered with uncertainty in system parameters. This problem is based on an F-16 aircraft model, for which a feedback control K has been designed for the nominal system. We wish to verify the robustness of the controller in the presence of parametric uncertainty in the F-16 model. For simplicity, it is assumed that the variation in the system parameters are dependent on a single random variable Δ (i.e., the variation in these parameters is not independent). In general, these parameters could be independent random processes. In this example, the short-period approximation of an F-16 is considered. The model is given by [24]

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

where the state vector $x = [\alpha \quad q \quad x_e]^T$; α is the angle of attack, q is the pitch rate, and x_e is an elevator state that captures actuator dynamics. The control $u = \delta_{ec}$ is the elevator command in degrees. The matrix parameters are

$$A = \begin{bmatrix} -0.6398 & 0.9378 & -0.0014 \\ (-1.5679) & (-0.8791) & (-0.1137) \\ 0 & 0 & -20.2000 \end{bmatrix}$$

$$B = [0 \quad 0 \quad 20.2]^T$$

$$C = \begin{bmatrix} 0 & \frac{180}{\pi} & 0 \end{bmatrix}$$

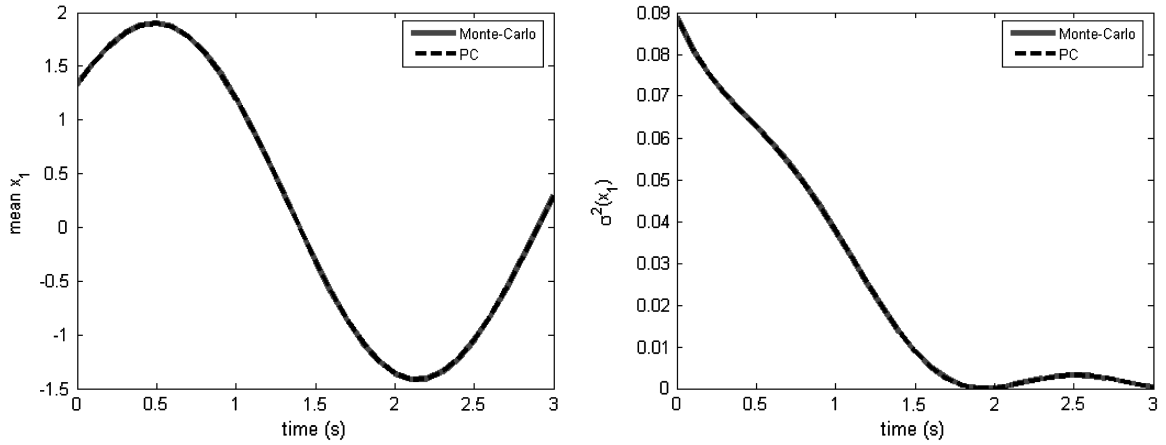


Fig. 4 Predicted mean and variance for a linear system with an initial condition uncertainty governed by a beta distribution using MC and GPC methods.

The values in parentheses are assumed to be uniformly distributed with a 10% deviation about their nominal values. This uncertainty is due to the uncertainty in the damping term C_{xq}, which is difficult to model in a high angle of attack [24]. A frequency domain control has been designed based on feedback of q for the nominal system. The control is of the form

$$u = \frac{0.3122s + 0.5538}{s^2 + 2.128s + 1.132} q$$

which is designed to be a pitch rate tracking controller. This is converted to the state-space form (A_c, B_c, C_c) and augmented to the system to arrive at the closed loop system:

$$\dot{x}_a = A_{\text{cl}} x_a + B_{\text{cl}} u$$

where

$$A_{\text{cl}} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}; \qquad B_{\text{cl}} = \begin{bmatrix} \mathbf{0} \\ B_c \end{bmatrix}$$

The accuracy of the GPC-based approach for finite dimensional approximation of linear stochastic dynamics can be inferred from Fig. 5. The circles represent the eigenvalues of the GPC system with ten terms. The solid dots represent the eigenvalues of the system obtained by sampling the stochastic system over Δ. It is interesting to note that the distribution of eigenvalues of the stochastic system is accurately captured by the eigenvalues of the GPC system. This gives us confidence in the use of PC for stability analysis and control of stochastic dynamical systems.

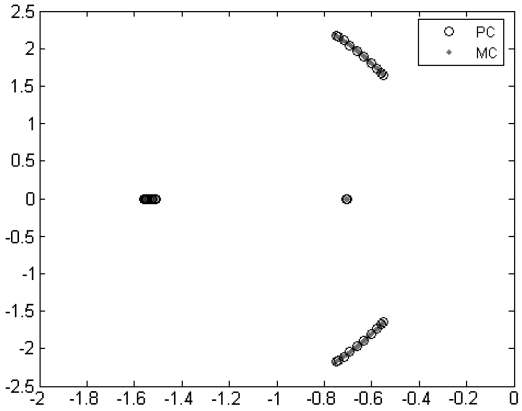


Fig. 5 Closed loop eigenvalue distributions for a short-period mode for ±20% parameter uncertainty.

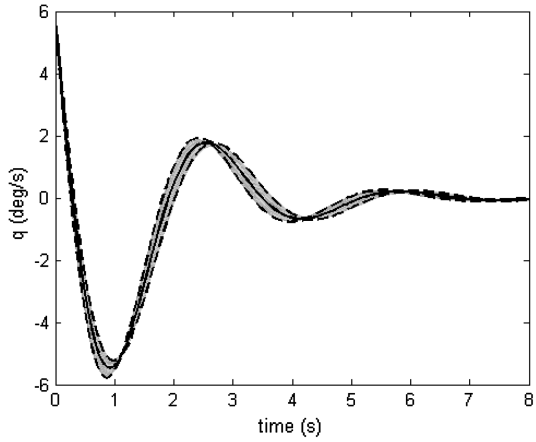


Fig. 6 Predicted and MC system response to ±10% parameter uncertainty.

Furthermore, we are able to understand how the uncertainty in system trajectories evolves over time. Figure 6 shows the pitch rate response of the system in the presence of ±10% system uncertainty in the aforementioned parameters. The predicted mean and trajectory bounds from GPC are represented by the dark solid and dashed lines, respectively. The MC responses of each system are depicted in gray. We observe that the bounds predicted by the GPC system are in excellent agreement with the responses of the MC simulations. In this manner, we are able to predict the statistical behavior of the system through examination of the GPC system, which is computationally more efficient than MC methods.

IV. Vinh's Equation with Probabilistic System Parameters

In this section, we use PC framework for analyzing uncertainty propagation in hypersonic flight vehicles. Here, we consider the three degrees of freedom in Vinh's equation to model hypersonic flight dynamics; the state variables are height h (from the surface of the planet), velocity v, and flight path angle γ. The equation of motion in the longitudinal plane is given by

$$\left.\begin{aligned} \dot{h} &= v \sin\gamma, \\ \dot{v} &= -\frac{\rho_0}{2B_c} \exp\left(\frac{h_2 - hR_0}{h_1}\right) v^2 - \sin\gamma, \\ \dot{\gamma} &= \cos\gamma \left(\frac{v^2 - 1}{v}\right) + \frac{\rho_0 \nu}{2B_c} \exp\left(\frac{h_2 - hR_0}{h_1}\right) v \end{aligned}\right\} \tag{20}$$

where R_0, h_1, h_2, B_c, ρ_0, and ν are constants defined in the Nomenclature.

Parametric uncertainty is assumed to be in ρ_0 (density on the surface of Mars), ν (lift over drag ratio), and B_c (ballistic coefficient). The uncertainty in these parameters is assumed to be uniform. Therefore, from Table 1, the basis functions are given by Legendre polynomials. The random variables ν, B_c, and ρ_0 can then be written as

$$\begin{aligned} \nu(\Delta) &= \bar{\nu} + \delta\nu\Delta; \\ B_c(\Delta) &= \bar{B}_c + \delta B_c \Delta; \\ \rho_0(\Delta) &= \bar{\rho}_0 + \delta\rho_0\Delta \end{aligned}$$

where $\Delta \in [-1, 1]$ and $\delta\nu$, δB_c, and $\delta\rho_0$ are the perturbations about the nominal values $\bar{\nu}$, $\bar{B}_c$, and $\bar{\rho}_0$, respectively. These parameters are assumed to have uniform distributions about their nominal values. When ν, B_c, and ρ_0 are random variables, the differential equation defined by Eq. (20) is stochastic, and h, v, and γ are random processes. PC framework is used to transform the stochastic differential equation into a deterministic differential equation in higher dimension state space. The random processes $h(t, \Delta)$, $v(t, \Delta)$, and $\gamma(t, \Delta)$ are expanded as

$$\begin{aligned} h(t, \Delta) &= \sum_0^P h_i(t)\phi_i(\Delta); \\ v(t, \Delta) &= \sum_0^P v_i(t)\phi_i(\Delta); \\ \gamma(t, \Delta) &= \sum_0^P \gamma_i(t)\phi_i(\Delta) \end{aligned}$$

Note that for the parameters with uniform uncertainty, only two basis functions are required to completely capture their respective probability density functions. No benefit is obtained by including more terms. However, for the states, the expansion includes higher order basis functions. From the GPC theory, exponential convergence is guaranteed as higher order basis functions are included [16].

Substituting these in Eq. (20) results in the following:

$$\sum_0^P \dot{h}_i\phi_i = \sum_0^P v_i\phi_i \sin\left(\sum_0^P \gamma_i\phi_i\right);$$

$$\sum_0^P \dot{v}_i\phi_i = -\frac{\rho_0(\Delta)}{2B_c(\Delta)}\left[\left(\exp\frac{h_2}{h_1}\right)/\left(\exp\frac{R_0\sum_0^P h_i\phi_i}{h_1}\right)\right] \times \sum_0^P\sum_0^P \phi_i\phi_j v_i v_j - \sin\left(\sum_0^P \gamma_i\phi_i\right);$$

$$\sum_0^P \dot{\gamma}_i\phi_i = \cos\left(\sum_0^P \gamma_i\phi_i\right)\left\{\sum_0^P v_i\phi_i - \left[1/\left(\sum_0^P v_i\phi_i\right)\right]\right\} + \frac{\rho_0(\Delta)\nu(\Delta)}{2B_c(\Delta)}\left[\left(\exp\frac{h_2}{h_1}\right)/\left(\exp\frac{R_0\sum_0^P h_i\phi_i}{h_1}\right)\right]\sum_0^P v_i\phi_i$$

The preceding equations are further simplified by the following substitutions:

$$\sum_0^P w_i\phi_i = \left[1/\left(\sum_0^P v_i\phi_i\right)\right];$$

$$\sum_0^P x_i\phi_i = \left[\left(\exp\frac{h_2}{h_1}\right)/\left(\exp\frac{R_0\sum_0^P h_i\phi_i}{h_1}\right)\right];$$

$$\sum_0^P y_i\phi_i = \frac{\rho_0(\Delta)}{2B_c(\Delta)}; \qquad \sum_0^P z_i\phi_i = \frac{\rho_0(\Delta)\nu(\Delta)}{2B_c(\Delta)}$$

where coefficients w_i, x_i, y_i, and $z_i \in \mathbb{R}$ are yet to be determined. The Vinh's equation can now be written as

$$\sum_0^P \dot{h}_i\phi_i = \sum_0^P v_i\phi_i \sin\left(\sum_0^P \gamma_i\phi_i\right);$$

$$\sum_0^P \dot{v}_i\phi_i = -\sum_{i,j,k,l=0}^P \phi_i\phi_j\phi_k\phi_l x_i y_j v_k v_l - \sin\left(\sum_{i=0}^P \gamma_i\phi_i\right);$$

$$\sum_0^P \dot{\gamma}_i\phi_i = \cos\left(\sum_{i=0}^P \gamma_i\phi_i\right)\left(\sum_{i=0}^P v_i\phi_i - \sum_{i=0}^P w_i\phi_i\right) + \sum_{i,j,k=0}^P \phi_i\phi_j\phi_k x_i z_j v_k$$

Taking the Galerkin projection on basis functions $\phi_i(\Delta)$, we get the following deterministic differential equations:

$$\left.\begin{aligned} \dot{h}_m &= \tfrac{1}{\langle\phi_m^2\rangle}\textstyle\sum_0^P v_i\langle\phi_i\phi_m \sin(\sum_0^P \gamma_i\phi_i)\rangle, \\ \dot{v}_m &= -\tfrac{1}{\langle\phi_m^2\rangle}\textstyle\sum_{i,j,k,l=0}^P\langle\phi_i\phi_j\phi_k\phi_l\phi_m\rangle x_i y_j v_k v_l - \tfrac{1}{\langle\phi_m^2\rangle}\langle\phi_m \sin(\textstyle\sum_{i=0}^P \gamma_i\phi_i)\rangle, \\ \dot{\gamma}_m &= \tfrac{1}{\langle\phi_m^2\rangle}\textstyle\sum_{i=0}^P v_i\langle\phi_i\phi_m\cos(\sum_{i=0}^P \gamma_i\phi_i)\rangle - \tfrac{1}{\langle\phi_m^2\rangle}\sum_{i=0}^P w_i\langle\phi_i\phi_m\cos(\sum_{i=0}^P \gamma_i\phi_i)\rangle + \tfrac{1}{\langle\phi_m^2\rangle}\sum_{i,j,k=0}^P\langle\phi_i\phi_j\phi_k\phi_m\rangle x_i z_j v_k \end{aligned}\right\} \tag{21}$$

where $m = 0, \cdots, P$. Equation (21) is the equivalent deterministic dynamics of the stochastic dynamics given by Eq. (20), approximated by GPC expansions. The solution of this differential equation, in higher dimensional state space, can then be used to characterize $h(t, \Delta)$, $v(t, \Delta)$, and $\gamma(t, \Delta)$.

The terms w_i, x_i, y_i, and z_i in Eq. (21) are computed as follows. Define

$$\alpha = \sum_0^P \gamma_i\phi_i; \qquad \beta = \left[\left(R_0\sum_0^P h_i\phi_i\right)/h_1\right]$$

Multiplying

$$\sum_0^P x_i\phi_i = \left[\left(\exp\frac{h_2}{h_1}\right)/\left(\exp\frac{R_0\sum_0^P h_i\phi_i}{h_1}\right)\right]$$

by $\exp(\beta)$ on both sides, we get

$$\exp\frac{h_2}{h_1} = \sum_0^P x_i\phi_i \exp\beta$$

Taking projection on the basis functions yields

$$\left\langle\exp\frac{h_2}{h_1}\phi_k\right\rangle = \left\langle\sum_0^P \phi_i\phi_k\exp\beta\right\rangle x_i$$

This produces a set of $(P+1)$ linear equations in x_i. Similarly, w_i are computed by multiplying

$$\left[1/\left(\sum_0^P v_i\right)\right]\phi_i = \sum_0^P w_i\phi_i$$

by

$$\sum_0^P v_i\phi_i$$

on both sides, which reduces to

$$1 = \sum_0^P\sum_0^P \phi_i\phi_j v_i w_j$$

Now, taking a projection on the basis functions, we get

$$\langle\phi_k\rangle = \sum_0^P\sum_0^P\langle\phi_i\phi_j\phi_k\rangle v_i w_j$$

which is also a system of linear equations in w_i. The coefficients y_i and z_i can also be determined in a similar manner.

The terms $\sin(\alpha)$ and $\exp(\beta)$ are computed numerically. These terms can also be computed by approximating them by using the Taylor series expansion of the perturbation about the mean [25]. For example, computing the exponential of a random variable ξ, the perturbation around the mean is given as $d = \xi - \xi_0$, where ξ_0 is the mean. Therefore,

$$\exp(\xi) = \exp(\xi_0)\left(1 + \sum_{n=1}^{N_{\text{tay}}}\frac{d^n}{n!}\right)$$

Here, d is the stochastic part of ξ, and N_{tay} is the number of terms in the Taylor series expansion. The sine of the random process is computed, similarly to the exponential, in the following manner:

$$\sin(\xi) = \sin(\xi_0 + d) = \sin(\xi_0)\cos(d) + \cos(\xi_0)\sin(d)$$

Although the Taylor series approximation is straightforward and generally computationally cost effective, it becomes severely

inaccurate when higher order GPC expansions are required to represent the physical variability [25]. For example, a fifth-order Taylor series approximation using third-order GPC expansion will require tensor products of six third-order basis functions. This will result in 18th-order polynomials. This will increase if higher order Taylor series or GPC expansions are used to obtain better approximations. It is well known that the computation of higher order polynomials using finite bit representation of real numbers has associated numerical errors. At the same time, for many nonlinear functions, this Taylor series approximation is limited by the theoretical range of convergence Taylor series. To tackle the problem of inaccuracies in the evaluation of transcendental functions, using Taylor series expansions, a more robust algorithm is presented by Debusschere et al. [25]. This method is valid for any nonpolynomial function $u(\xi)$ for which $\frac{du}{d\xi}$ can be expressed as a rational function of ξ, $u(\xi)$. This approach has not been applied in the research presented in this paper.

V. Obtaining Statistics from Polynomial Chaos

Mean and variance of the state trajectories can be easily computed from the coefficients of the GPC expansions. The mean trajectories can be derived as

$$\mathbf{E}\left[h(t,\ \Delta)\right] = \mathbf{E}\left[\sum_{i=0}^{p} h_i \phi_i\right] = \sum_{i=0}^{p} h_i \mathbf{E}[\phi_i] = \sum_{i=0}^{p} h_i \int_{\mathcal{D}_\Delta} \phi_i f \, \mathrm{d}\Delta$$

and similarly

$$\mathbf{E}\left[v(t,\ \Delta)\right] = \sum_{i=0}^{p} v_i \int_{\mathcal{D}_\Delta} \phi_i f \, \mathrm{d}\Delta;$$

$$\mathbf{E}[\gamma(t,\ \Delta)] = \sum_{i=0}^{p} \gamma_i \int_{\mathcal{D}_\Delta} \phi_i f \, \mathrm{d}\Delta$$

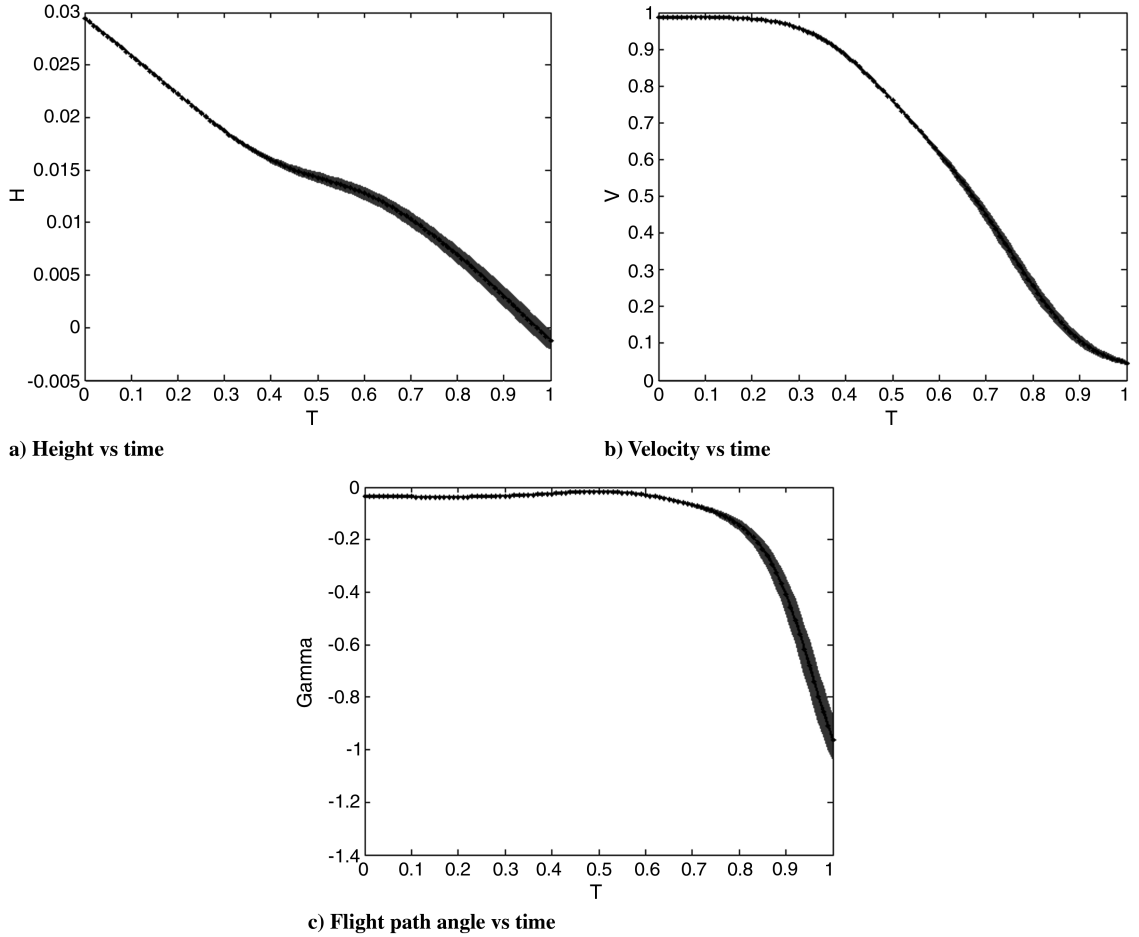
where h_i, v_i, and γ_i are the coefficients of the GPC expansions of $h(t,\ \Delta)$, $v(t,\ \Delta)$, and $\gamma(t,\ \Delta)$ respectively, and f is the probability density function of the parameters.

The variance of a trajectory $h(t,\ \Delta)$ is given by

$$\begin{aligned}\sigma^2[h(t,\ \Delta)] &= \mathbf{E}[h(t,\ \Delta)^2] - \mathbf{E}[h(t,\Delta)]^2; \\ &= \sum_{i,j=0}^{P} h_i h_j \int_{\mathcal{D}_\Delta} \phi_j \phi_j f \ \mathrm{d}\Delta - \mathbf{E}[h(t,\ \Delta)]^2 \\ &= \sum_{i=0}^{P} h_i^2 \int_{\mathcal{D}_\Delta} \phi_i^2 f \ \mathrm{d}\Delta - \mathbf{E}[h(t,\ \Delta)]^2\end{aligned}$$

because of the orthogonality of ϕ_i and ϕ_j. In this manner, the covariance matrix for the system can also be determined.

In the preceding computation, the probability density function for Δ is time invariant because we assume that the statistical characteristics of the parameters do not change with time (i.e., they are random variables with stationary distributions). However, the probability density function associated with the state trajectory $x(t,\ \Delta)$ (state vector of the stochastic dynamical system) will change with time. In the PC framework, this density function cannot be determined explicitly, except under Gaussian assumptions [12]. However, the time-varying moments corresponding to the time-varying

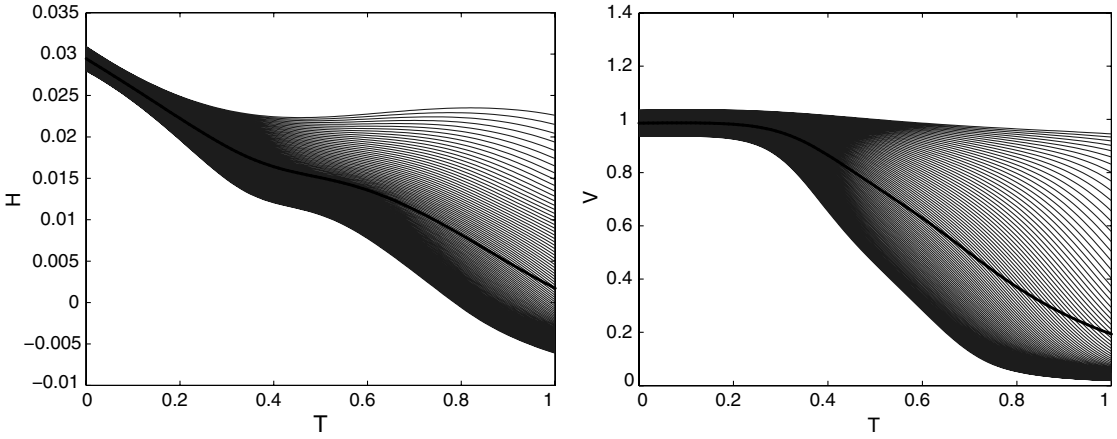


a) Height vs time

b) Velocity vs time

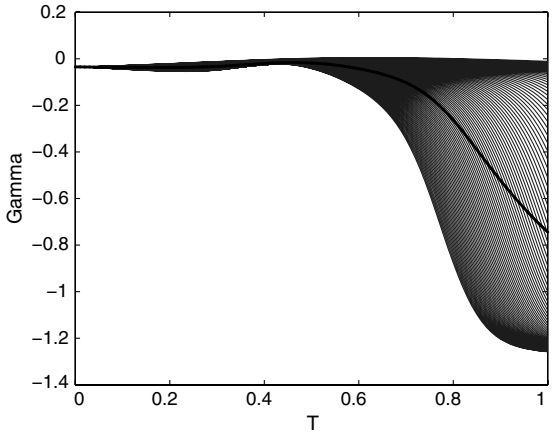
c) Flight path angle vs time

Fig. 7 Evolution of uncertainty with 5% uniform uncertainty in v, B_c, and ρ_0.



a) Height vs time

b) Velocity vs time



c) Flight path angle vs time

Fig. 8 Evolution of uncertainty with 5% uniform uncertainty in the initial conditions.

probability density function of $x(t,\ \Delta)$ can be determined from time-varying GPC coefficients of $x(t,\ \Delta)$ and the fixed probability density function of Δ, as shown in the preceding expressions.

VI. Verification Using Monte–Carlo Simulations

A. Uniform Distribution

In this section, we analyze the effect of uncertainty in the initial conditions and parameters on hypersonic flight trajectories, and we compare the results obtained from MC simulations and PC theory. We have assumed 10% parametric variation in initial conditions (h_0, v_0, and γ_0) and parameters (ρ_0, ν, and Bc) about the nominal values. The nominal values for ($h_0,\ v_0$, and γ_0) are taken as (100 km, 3.5 km/s, and -2 deg), respectively. The nominal values for (ρ_0, ν, and Bc) are listed in listed in the Nomenclature. The distribution is assumed to be uniform about the nominal values. To identify the sensitivity of the state trajectories to these uncertain parameters, analysis has been performed with one single parameter uncertainty at a time. A coupled analysis can also be performed by considering all the parameters to be uncertain at the same time. Statistics obtained from MC simulations have been compared with those obtained from PC theory. The results match very well and are obtained with significantly reduced computational overhead. We used 30 sample points, for each parameter, for the MC simulations.

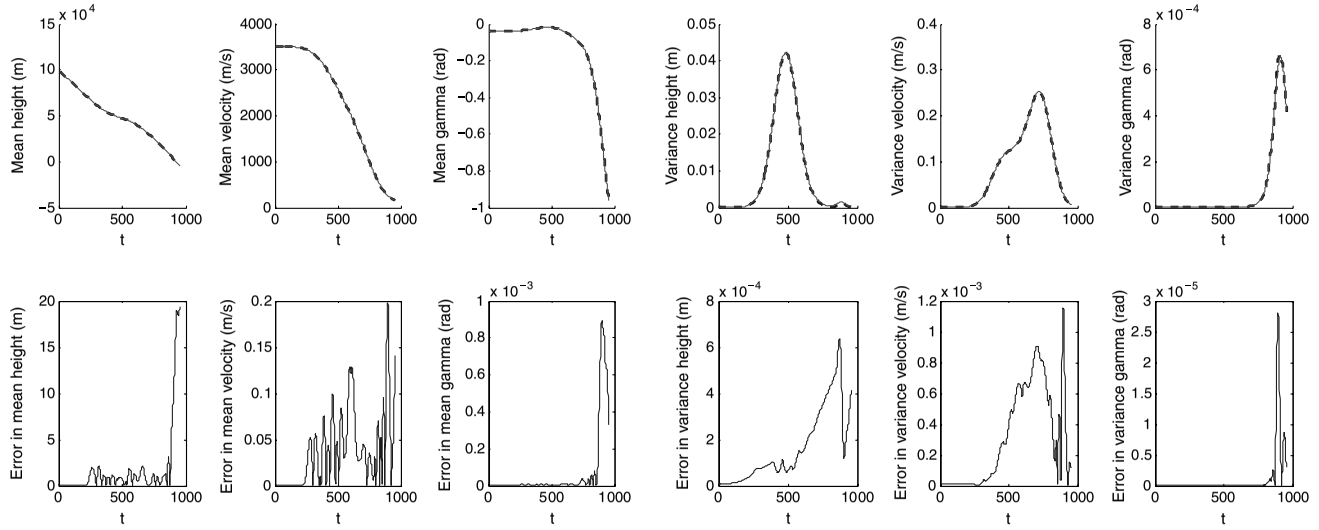
Figure 7 shows the evolution of uncertainty in state trajectories due to parametric uncertainty. It can be seen that the effect of uncertainty in ρ_0, ν, and Bc are pronounced in the lower altitudes, where the aerodynamic effects are dominant. Figure 8 shows the results for uncertainty in initial conditions and reveals that the state trajectories of the vehicle are greatly affected by the uncertainty in the initial condition. These trajectories were obtained using MC simulations.

The sensitivity of the state trajectories with respect to the previously mentioned parametric uncertainty is analyzed next. Separate analysis to determine the effect of uncertainty in the parameters Bc, ν, and ρ_0 is performed. Figures 9–11, show the trajectories obtained in each of these cases (using PC theory and MC simulations). These plots show very little error between the results obtained from these two methods. In terms of the sensitivity of the trajectories to these parameter variations, it can be seen that the mean trajectories are almost identical for the various parametric uncertainty considered. There is, however, a significant difference in the variance trajectories.

Figure 12 shows the convergence of error in the statistics obtained from the MC method and the PC theory. Here, the convergence properties of PC can be seen, as better results are obtained for higher order approximations.

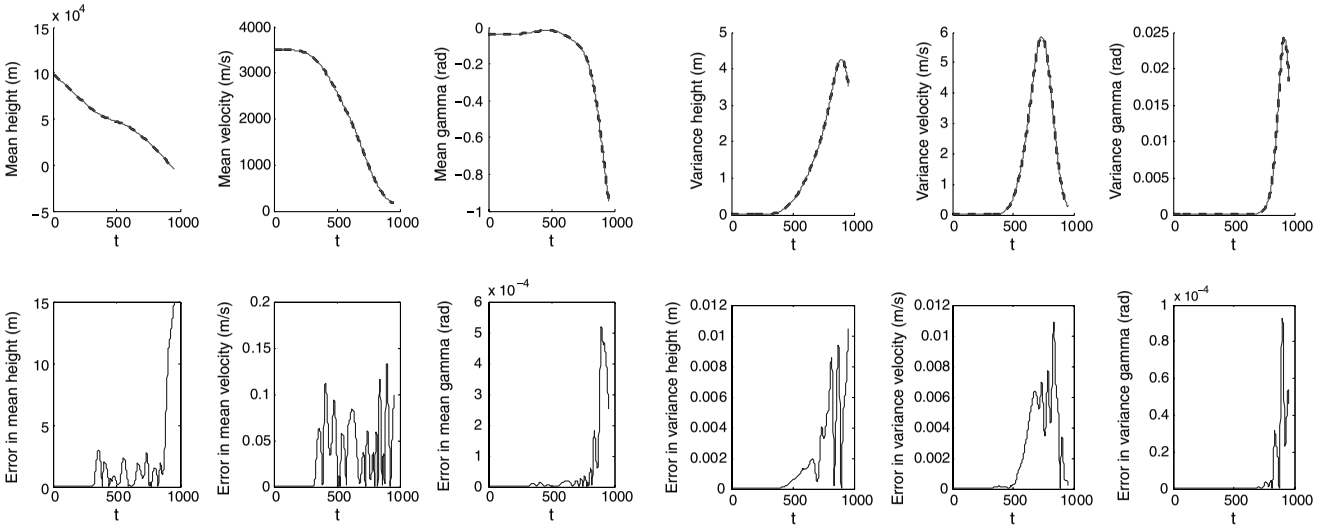
B. Gaussian Distribution

Here, the initial condition is assumed to be governed by Gaussian distribution, and its effect on state trajectories are obtained and compared using MC simulations, PC theory, and uncertainty propagation via linearization of Vinh's equation and techniques described in Sec. III. For this example, only uncertainty in height is assumed. The uncertainty is assumed to be Gaussian with a mean of 100 km and $\sigma = 5.82$ km. It can be seen from Figs. 13 and 14 that PC has less error than results obtained from linear analysis when compared with MC simulations. The numerical simulations were performed within $\pm 5\sigma$ of the Gaussian distribution. A common method to propagate Gaussian uncertainty for nonlinear systems is to



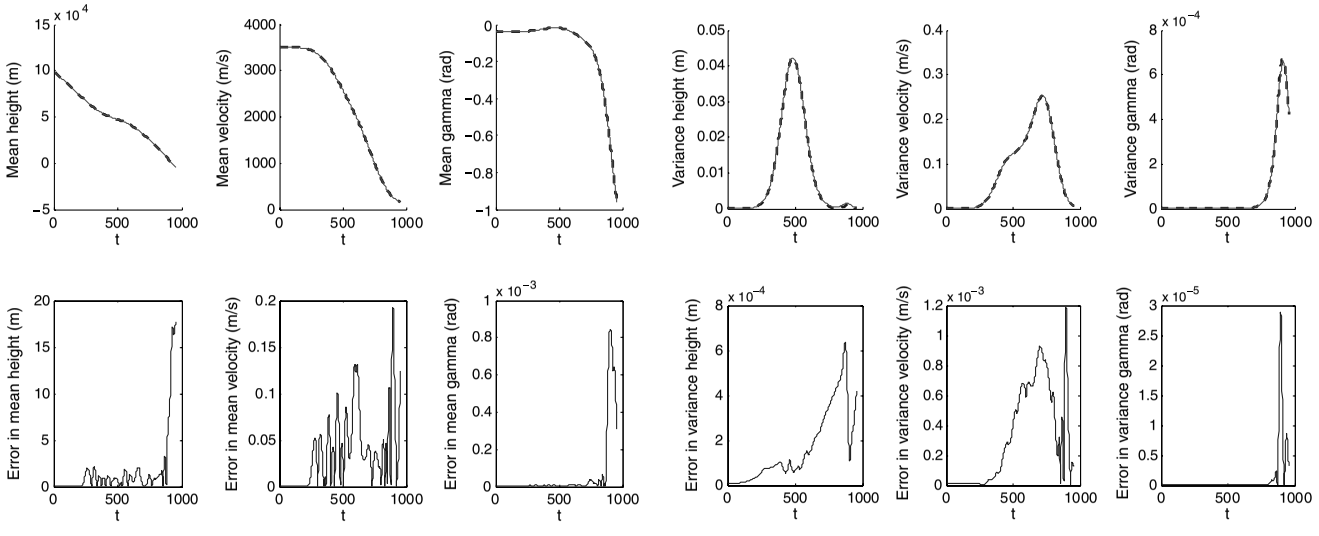
a) Mean trajectories **b) Variance trajectories**

Fig. 9 10% uncertainty in B_c: MC method (dashed) and PC method (solid).



a) Mean trajectories **b) Variance trajectories**

Fig. 10 10% uncertainty in v: MC method (dashed) and PC method (solid).



a) Mean trajectories **b) Variance trajectories**

Fig. 11 10% uncertainty in ρ_0: MC method (dashed) and PC method (solid).

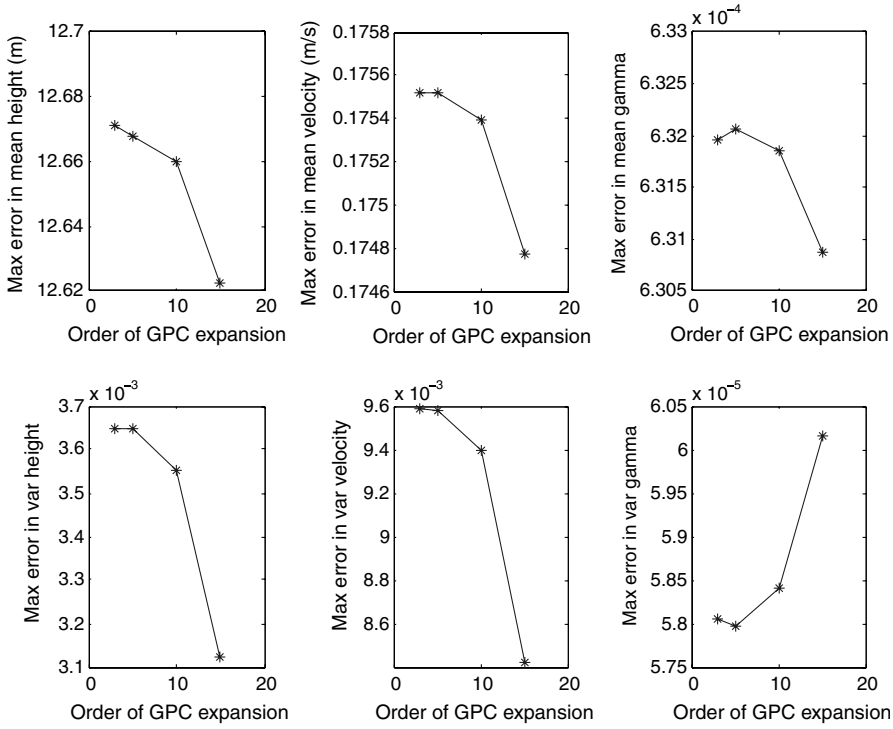
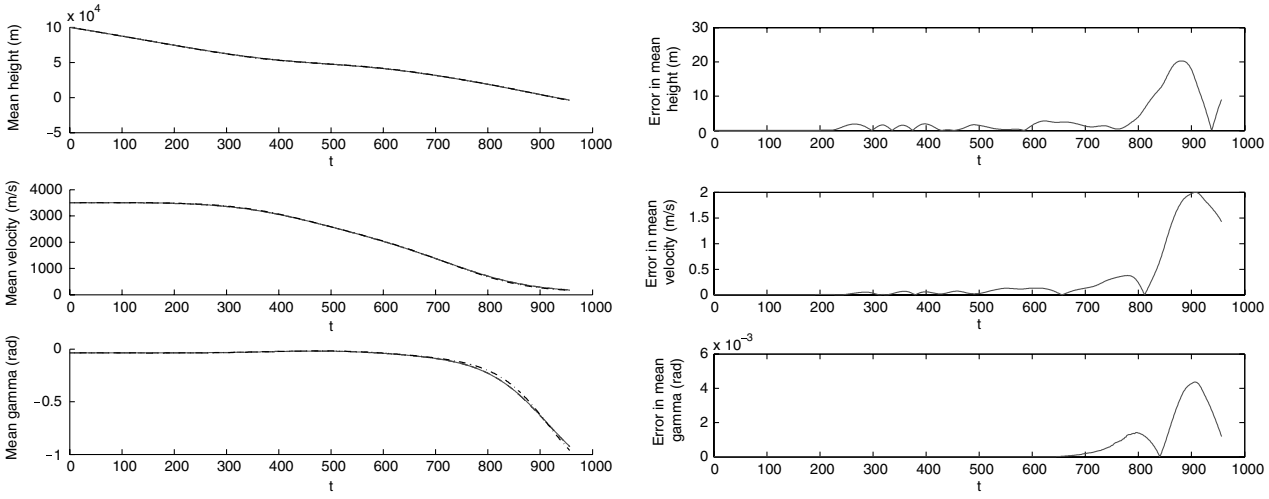
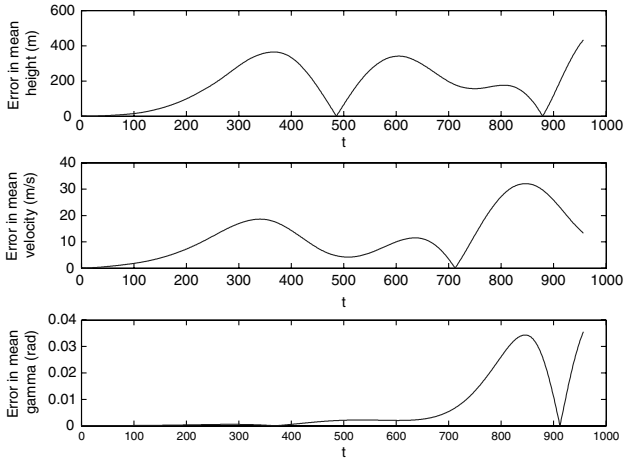


Fig. 12 Error convergence with order of PC expansion.



a) Mean trajectory

b) Error in mean between GPC and Monte-Carlo



c) Error in mean between linear analysis and Monte-Carlo

Fig. 13 Comparison of results obtained from linear analysis and PC theory. Uncertainty is assumed to be Gaussian in height with the mean = 100 km and $\sigma = 5.82$ km.

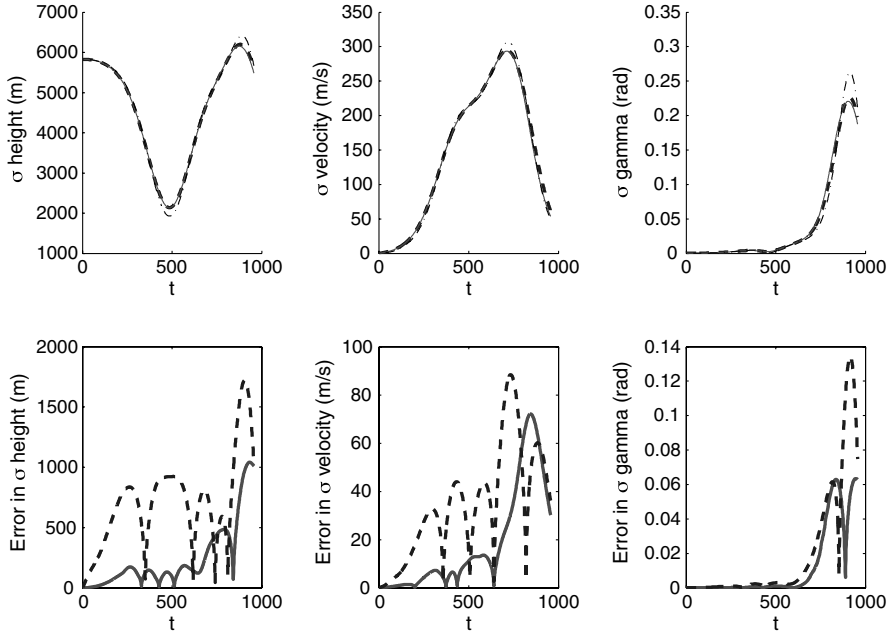


Fig. 14 Variance from the MC method (dashed), the PC theory (solid), linear analysis (dash-dot), and the associated error when compared with the MC method (with GPC method: solid, with linear analysis: dashed). Uncertainty is assumed to be in the initial height with the mean = 100 km and $\sigma = 5.82$ km.

linearize the system about the mean trajectory and use methods similar to EKF to propagate the uncertainty. Such methods are known to perform poorly when the nonlinearities are dominant. Better results are obtained with PC theory.

VII. Computational Time

In [26], Prabhakar and Bhattacharya had reported that MC simulations took roughly 116.12 s and PC took about 2 s. These computational times were for cases with one uncertain parameter. When rewriting the code for the MC simulations, via the vectorization of the MATLAB® code, we were able to reduce the computational time to about 14 s. However, even with the improvements in the MC simulation code, the PC approach was clearly more efficient computationally. For higher dimensional parameter space, this advantage was even more significant. For example, with three uncertain parameters, MC simulations took 372.2 s and PC took 16.9 s. The computational times reported for PC included the one time computation of the inner products mentioned in the problem formulation. Excluding them would reduce the computational time further. The computational time in that case would be the time taken to integrate the augmented state dynamics.

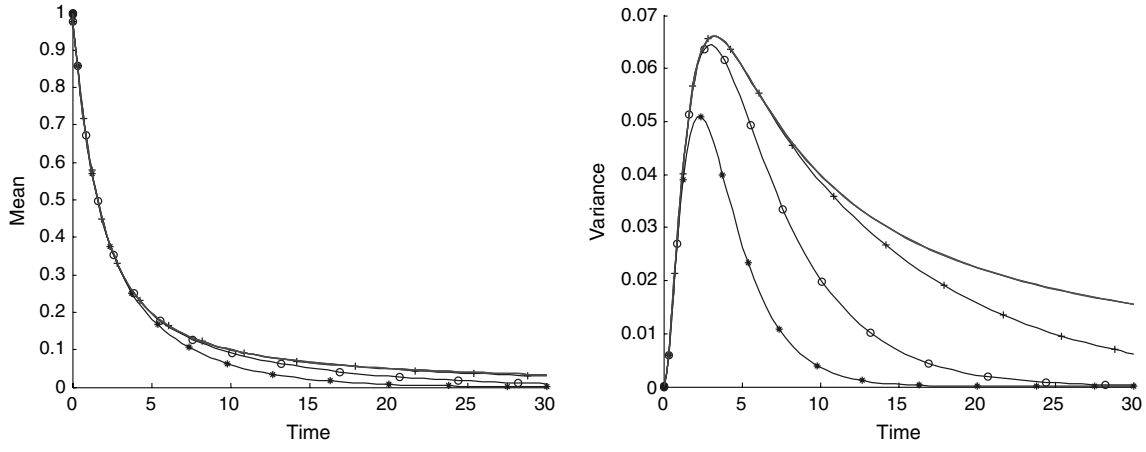
VIII. Limitations of Polynomial Chaos

The GPC framework is well suited for evaluating short-term statistics of dynamical systems, as the accuracy of the estimates degrades over time. This degradation arises due to finite dimensional approximation of the probability space $(\Omega, \mathcal{F}, P)$. Figure 15 shows the error in mean and variance for a one-dimensional linear system. The dynamics is given by $\dot{x} = -ax$, with a uniformly distributed over [0, 1]. For such a system, the analytical expression for mean and variance is $\bar{x}(t) = \frac{1-e^{-t}}{t}$ and

$$\sigma(t) = \frac{1 - e^{-2t}}{2t} - \left(\frac{1 - e^{-t}}{t}\right)^2$$

In Fig. 15, we observe that the estimates of mean and variance from GPC get closer to the analytical values for higher order GPC expansions. We also observe that for a given order of expansion, lower order moments have higher accuracy than higher order moments. Other examples of such deviations are available in [27].

Several methods have been proposed to reduce this degradation, including adaptive [28] and multi-element approximation techniques [29], and these will be incorporated in our future work.



a) Mean **b) Variance**

Fig. 15 Errors in estimates obtained from the GPC method for $\dot{x} = -a(\Delta)x$: analytical (solid), GPC method: second order (*), third order (o), and fifth order (+).

IX. Conclusions

In this paper, a framework for analyzing the evolution of uncertainty in hypersonic flight is presented. This framework is based on the GPC framework, which has been successfully applied to areas such as stochastic computational fluid dynamics and solid mechanics. It has been demonstrated that this framework captures the trajectory statistics quite accurately and is computationally less demanding than methods based on MC simulations. MC methods were used in this paper solely to verify the results obtained from the GPC approach.

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